

ON FERMIONIC REPRESENTATION OF THE GROMOV-WITTEN INVARIANTS OF THE RESOLVED CONIFOLD

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ABSTRACT. We prove that the fermionic form of the generating function of the Gromov-Witten invariants of the resolved conifold is a Bogoliubov transform of the fermionic vacuum; in particular, it is a tau function of the KP hierarchy. Our proof is based on the gluing rule of the topological vertex and the formulas of the fermionic representations of the framed one-legged and two-legged topological vertex which were conjectured by Aganagic et al and proved in our recent work.

1. INTRODUCTION

In general it is an unsolved problem to compute the Gromov-Witten invariants of an algebraic variety in arbitrary genera. However, in the case of toric Calabi-Yau threefolds (which are noncompact), string theorists have found an algorithm called the topological vertex [2] to compute the generating function of both open and closed Gromov-Witten invariants based on a remarkable duality with link invariants in the Chern-Simons theory approach of Witten [17, 18]. A mathematical theory of the topological vertex has been developed in [12].

The topological vertex, which is the generating function of the Gromov-Witten invariants of \mathbb{C}^3 with three special D -branes, is a mysterious combinatorial object that asks for further studies. On the A -theory side, the topological vertex can be realized as a state in the threefold tensor product of the space Λ of symmetric functions. In this representation its expressions given by physicists [2] or by mathematicians [12] are both very complicated. It is very interesting to understand the topological vertex from other perspectives. In [15], the topological vertex is related to a combinatorial problem of plane partitions. In [2] it was suggested that the topological vertex is a Bogoliubov transform via the boson-fermion correspondence. This point of view was further elaborated in [1] and extended to the partition functions of toric Calabi-Yau threefolds. Indeed, by the local mirror symmetry [8, 7], on the B -model side, one studies quantum Kodaira-Spencer theory of the local mirror curve. By physical derivations, the corresponding state is constrained by the Ward identities, giving the W_∞ constraints. (See also [5] where the partition functions are expected to be annihilated by certain quantum operators obtaining by quantized the local mirror curves.) In this formalism it is natural to use the fermionic picture, and a simple looking formula for the fermionic form of the topological vertex under the boson-fermion correspondence was conjectured in [1]. The ADKMV conjecture is directly related to integrable hierarchies: The one-legged case is related to the KP hierarchy, the two-legged case to the 2-dimensional Toda hierarchy, and the three-legged case to the 3-component KP hierarchy (see

Remark 2.1). The one-legged and the two-legged cases can also be seen directly from the bosonic picture [19], but the three-legged case can only be seen through the fermionic picture.

The topological vertex can be used to compute Gromov-Witten invariants of toric Calabi-Yau 3-folds by certain gluing rules. There is a standard inner product on the space Λ , and the gluing rule is essentially taking inner product over the components corresponding to the branes of gluing (see §3.1 for exact formulation). So the resulted generating functions are states in multifold tensor products of the space Λ . In general, they have very complicated combinatorial structures.

In our recent work [3], we proposed a generalization of the ADKMV conjecture to the framed topological vertex which we refer to as the framed ADKMV conjecture. Note that it is important to consider framing when we consider gluing of the topological vertex. We gave a proof in [3] of the framed ADKMV conjecture in the one-legged case and the two-legged case, and derived a determinantal formula for the framed topological vertex in the three-legged case based on the Framed ADKMV Conjecture. It remains open to give a proof of this conjecture for the full three-legged topological vertex.

Provided that the framed ADKMV conjecture holds, then a natural question is whether or not the generating functions of the Gromov-Witten invariants of general toric Calabi-Yau threefolds are Bogoliubov transforms in the fermionic picture. It was also conjectured in [1] that it is indeed the case. However, it seems very difficult to prove this conjecture directly by boson-fermion correspondence and standard Schur calculus, even for the very simple case of the resolved conifold with a single brane. In [16] the closed string partition function of the resolved conifold is related to Hall-Littlewood functions and a fermionic representation is obtained by the deformed boson-fermion correspondence. Based on the method in [1], it was shown in [11] that the B-model amplitude of the mirror space of the one-legged resolved conifold is a Bogoliubov transform. It also seems difficult to generalize the method in [1] and [11] to prove this conjecture in general.

In this paper we will tackle this problem using a different strategy. We will start from the framed ADKMV conjecture, and then consider the gluing rule of the topological vertex as presented in [2][12] in the fermionic picture. In this work we will focus on the framed one-legged resolved conifold. But the method here can be easily modified to the cases of the total spaces of vector bundles $O(p) \oplus O(-2 - p) \rightarrow \mathbb{P}^1$, $p \in \mathbb{Z}$. The treatment for general toric Calabi-Yau threefolds will be presented in a separate paper [4]. The main result of the present paper is that the generating function of the Gromov-Witten invariants of the resolved conifold with one brane and arbitrary framing is a Bogoliubov transform of the fermionic vacuum; in particular, it is a tau function of the KP hierarchy.

The rest of the paper is arranged as follows. After reviewing some preliminaries and fixing notations in §2, we rewrite in §3 the generating function of Gromov-Witten invariants of the framed one-legged resolved conifold as an gluing (see that section for precise meaning) of two fermionic states which are Bogoliubov transforms, based on the fermionic representation of the framed one-legged and two-legged topological vertex. In the final §4, we prove that the gluing of an arbitrary two-component Bogoliubov transform and an arbitrary one-component Bogoliubov transform is also a Bogoliubov transform. The result in §4, combining with §3, leads

directly to the result that the fermionic representation of the generating function considered is a Bogoliubov transform.

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2. PRELIMINARIES

In this section, we recall briefly some well-known concepts and results that will be used in following sections.

2.1. Partitions. A partition μ of a positive integral number n is a decreasing finite sequence of integers $\mu_1 \geq \dots \geq \mu_l > 0$, such that $|\mu| = \mu_1 + \dots + \mu_l = n$. The following number associated to μ will be useful in this paper:

$$(1) \quad \kappa_\mu = \sum_{i=1}^l \mu_i(\mu_i - 2i + 1).$$

It is very useful to graphically represent a partition by its Young diagram. This leads to many natural definitions. First of all, by transposing the Young diagram one can define the conjugate μ^t of μ . Secondly assume the Young diagram of μ has k boxes in the diagonal. Define $m_i = \mu_i - i$ and $n_i = \mu_i^t - i$ for $i = 1, \dots, k$, then it is clear that $m_1 > \dots > m_k \geq 0$ and $n_1 > \dots > n_k \geq 0$. The partition μ is completely determined by the numbers m_i, n_i . We often denote the partition μ by $(m_1, \dots, m_k | n_1, \dots, n_k)$, this is called the Frobenius notation. A partition of the form $(m | n)$ in Frobenius form is called a hook partition.

For a box e at the position (i, j) in the Young diagram of μ , define its content by $c(e) = j - i$. Then it is easy to see that

$$(2) \quad \kappa_\mu = 2 \sum_{e \in \mu} c(e).$$

Indeed,

$$\sum_{e \in \mu} c(e) = \sum_{i=1}^l \sum_{j=1}^{\mu_i} (j - i) = \sum_{i=1}^n \left(\frac{1}{2} \mu_i(\mu_i + 1) - i \mu_i \right) = \frac{1}{2} \kappa_\mu.$$

A straightforward application of (2) is the following:

Lemma 2.1 (e.g see [3]). *Let $\mu = (m_1, m_2, \dots, m_k | n_1, n_2, \dots, n_k)$ be a partition written in the Frobenius notation. Then we have*

$$(3) \quad \kappa_\mu = \sum_{i=1}^k m_i(m_i + 1) - \sum_{i=1}^k n_i(n_i + 1).$$

In particular,

$$(4) \quad \kappa_{(m_1, m_2, \dots, m_k | n_1, n_2, \dots, n_k)} = \sum_{i=1}^k \kappa_{(m_i | n_i)}.$$

2.2. Schur functions and skew Schur functions. Let Λ be the space of symmetric functions in $\mathbf{x} = (x_1, x_2, \dots)$. The inner product on the space Λ is defined by setting the set of Schur functions as an orthonormal basis. For a partition μ , let $s_\mu := s_\mu(\mathbf{x})$ be the corresponding Schur function in Λ . Given to partitions μ and ν , the skew Schur functions $s_{\mu/\nu}$ is defined by the condition [13]

$$(s_{\mu/\nu}, s_\lambda) = (s_\mu, s_\nu s_\lambda)$$

for all partitions λ . This is equivalent to define

$$s_{\mu/\nu} = \sum_{\lambda} c_{\nu\lambda}^{\mu} s_{\lambda},$$

where the constants $c_{\nu\lambda}^{\mu}$ are the structure constants (called the Littlewood-Richardson coefficients) defined by

$$(5) \quad s_{\nu} s_{\lambda} = \sum_{\gamma} c_{\nu\lambda}^{\gamma} s_{\gamma}.$$

We often meet some specialization of symmetric functions. Let $q^\rho := (q^{-1/2}, q^{-3/2}, \dots)$. It is easy to see that

$$(6) \quad p_n(q^\rho) = \frac{1}{q^{n/2} - q^{-n/2}} = \frac{1}{[n]},$$

where $[n] = q^{n/2} - q^{-n/2}$. A very interesting fact is that with this specialization the Schur functions also have very simple expressions.

Proposition 2.2. [22] *For any partition μ , one has*

$$s_{\mu}(q^\rho) = q^{k_{\mu}/4} \frac{1}{\prod_{e \in \mu} [h(e)]},$$

where $h(e)$ is the hook number of e and $[n] = q^{n/2} - q^{-n/2}$.

2.3. Fermionic Fock space. We say a set of integers $A = \{a_1, a_2, \dots\} \subset \mathbb{Z} + \frac{1}{2}$, $a_1 > a_2 > \dots$, is admissible if it satisfies the following two conditions:

1. $\mathbb{Z}_- + \frac{1}{2} \setminus A$ is finite and
2. $A \setminus \mathbb{Z}_- + \frac{1}{2}$ is finite,

where \mathbb{Z}_- is the set of negative integers.

Consider the linear space W spanned by a basis $\{\underline{a} | a \in \mathbb{Z} + \frac{1}{2}\}$, indexed by half-integers. For an admissible set $A = \{a_1, a_2, \dots\}$, we associate an element $\underline{A} \in \wedge^\infty W$ as follows:

$$\underline{A} = \underline{a_1} \wedge \underline{a_2} \wedge \dots$$

Then the free fermionic Fock space \mathcal{F} is defined as

$$\mathcal{F} = \text{span}\{\underline{A} : A \subset \mathbb{Z} + \frac{1}{2} \text{ is admissible}\}.$$

One can define an inner product on \mathcal{F} by taking $\{\underline{A} : A \subset \mathbb{Z} + \frac{1}{2} \text{ is admissible}\}$ as an orthonormal basis.

For $\underline{A} = \underline{a_1} \wedge \underline{a_2} \wedge \dots \in \mathcal{F}$, define its charge as:

$$|A \setminus \mathbb{Z}_- + \frac{1}{2}| - |\mathbb{Z}_- + \frac{1}{2} \setminus A|.$$

Denote by $\mathcal{F}^{(n)} \subset \mathcal{F}$ the subspace spanned by \underline{A} of charge n , then there is a decomposition

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}^{(n)}.$$

An operator on \mathcal{F} is called charge 0 if it preserves the above decomposition.

The charge 0 subspace $\mathcal{F}^{(0)}$ has a basis indexed by partitions:

$$(7) \quad |\mu\rangle := \underline{\mu_1 - \frac{1}{2}} \wedge \underline{\mu_2 - \frac{3}{2}} \wedge \cdots \wedge \underline{\mu_l - \frac{2l-1}{2}} \wedge \underline{\frac{2l+1}{2}} \wedge \cdots$$

where $\mu = (\mu_1, \dots, \mu_l)$, i.e., $|\mu\rangle = \underline{A_\mu}$, where $A_\mu = (\mu_i - i + \frac{1}{2})_{i=1,2,\dots}$. If $\mu = (m_1, \dots, m_k | n_1, \dots, n_k)$ in Frobenius notation, then

$$(8) \quad |\mu\rangle = \underline{m_1 + \frac{1}{2}} \wedge \cdots \wedge \underline{m_k + \frac{1}{2}} \wedge \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \cdots \wedge \widehat{-n_k - \frac{1}{2}} \wedge \cdots \wedge \widehat{-n_1 - \frac{1}{2}} \wedge \cdots.$$

In particular, when μ is the empty partition, we get:

$$|0\rangle := -\frac{1}{2} \wedge -\frac{3}{2} \wedge \cdots \in \mathcal{F}.$$

It will be called the fermionic vacuum vector.

We now recall the creators and annihilators on \mathcal{F} . For $r \in \mathbb{Z} + \frac{1}{2}$, define operators ψ_r and ψ_r^* by

$$\begin{aligned} \psi_r(\underline{A}) &= \begin{cases} (-1)^k \underline{a_1} \wedge \cdots \wedge \underline{a_k} \wedge \underline{r} \wedge \underline{a_{k+1}} \wedge \cdots, & \text{if } a_k > r > a_{k+1} \text{ for some } k, \\ 0, & \text{otherwise;} \end{cases} \\ \psi_r^*(\underline{A}) &= \begin{cases} (-1)^{k+1} \underline{a_1} \wedge \cdots \wedge \widehat{\underline{a_k}} \wedge \cdots, & \text{if } a_k = r \text{ for some } k, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Under the inner product defined above, for $r \in \mathbb{Z} + 1/2$, it is clear that ψ_r and ψ_r^* are adjoint operators. The anti-commutation relations for these operators are

$$(9) \quad [\psi_r, \psi_s^*]_+ := \psi_r \psi_s^* + \psi_s^* \psi_r = \delta_{r,s} id$$

and other anti-commutation relations are zero. It is clear that for $r > 0$,

$$(10) \quad \psi_{-r}|0\rangle = 0, \quad \psi_r^*|0\rangle = 0,$$

so the operators $\{\psi_{-r}, \psi_r^*\}_{r>0}$ are called the fermionic annihilators. For a partition $\mu = (m_1, m_2, \dots, m_k | n_1, n_2, \dots, n_k)$, it is clear that

$$(11) \quad |\mu\rangle = (-1)^{n_1+n_2+\dots+n_k} \prod_{i=1}^k \psi_{m_i+\frac{1}{2}} \psi_{-n_i-\frac{1}{2}}^* |0\rangle.$$

So the operators $\{\psi_r, \psi_{-r}^*\}_{r>0}$ are called the fermionic creators. The normally ordered product is defined as

$$: \psi_r \psi_r^* := \begin{cases} \psi_r \psi_r^*, & r > 0, \\ -\psi_r^* \psi_r, & r < 0. \end{cases}$$

In other words, an annihilator is always put on the right of a creator.

2.4. Boson-fermion correspondence. For any integer n , define an operator α_n on the fermionic Fock space \mathcal{F} as follows:

$$\alpha_n = \sum_{r \in \mathbb{Z} + \frac{1}{2}} : \psi_r \psi_{r+n}^* :$$

Let $\mathcal{B} = \Lambda[z, z^{-1}]$ be the bosonic Fock space, where z is a formal variable. Then the boson-fermion correspondence is a linear isomorphism $\Phi : \mathcal{F} \rightarrow \mathcal{B}$ given by

$$(12) \quad u \mapsto z^m \langle \underline{0}_m | e^{\sum_{n=1}^{\infty} \frac{pn}{n} \alpha_n} u \rangle, \quad u \in \mathcal{F}^{(m)}$$

where $|\underline{0}_m\rangle = \frac{-1}{2} + m \wedge \frac{-3}{2} + m \wedge \dots$. It is clear that Φ induces an isomorphism between $\mathcal{F}^{(0)}$ and Λ . Explicitly, this isomorphism is given by

$$(13) \quad |\mu\rangle \longleftrightarrow s_\mu.$$

The boson-fermionic correspondence plays an important role in Kyoto school's theory of integrable hierarchies. For example,

Proposition 2.3. *If $\tau \in \Lambda$ corresponds to $|v\rangle \in F^{(0)}$, then τ is a tau-function of the KP hierarchy in the Miwa variable $t_n = \frac{pn}{n}$ if and only if $|v\rangle$ satisfies the bilinear relation*

$$(14) \quad \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r |v\rangle \otimes \psi_r^* |v\rangle = 0.$$

Remark 2.1. A state $|v\rangle \in \mathcal{F}^{(0)}$ satisfies the bilinear relation (14) if and only if it lies in the orbit $\widehat{GL_\infty}|0\rangle$. This is equivalent to say that $|v\rangle$ can be represented as

$$|v\rangle = \exp\left(\sum_{r,s \in \mathbb{Z} + 1/2} M_{rs} : \psi_r \psi_s^* : \right) |0\rangle$$

for some coefficients M_{rs} . There is also a multi-component generalization of the boson-fermion correspondence which can be used to study multi-component KP hierarchies [10].

3. GROMOV-WITTEN INVARIANTS OF THE FRAMED ONE-LEGGED RESOLVED CONIFOLD AND ITS FERMIONIC FORM

3.1. Generating function of Gromov-Witten invariants of the framed one-legged resolved conifold. By the theory of topological vertex [2][12], the Gromov-Witten invariants of any toric Calabi-Yau threefold can be computed from the topological vertex by certain explicit gluing process. As a special case, the Gromov-Witten invariants of the framed one-legged resolved conifold can be computed by gluing the framed one-legged topological vertex and the framed two-legged topological vertex.

The one-legged topological vertex with framing a , in terms of Schur functions, is given by

$$(15) \quad Z_1^{(a)}(\mathbf{y}) = \sum_{\mu} q^{a\kappa_\mu/2} s_\mu(q^\rho) s_\mu(\mathbf{y})$$

and the two-legged topological vertex with framings (a_1, a_2) , in terms of skew Schur functions, is given by [19] [22]

$$(16) \quad Z_2^{(a_1, a_2)}(\mathbf{x}; \mathbf{y}) = \sum_{\mu, \nu} \left(q^{\frac{(a_1+1)\kappa_\mu + a_2\kappa_\nu}{2}} \sum_{\eta} s_{\mu^t/\eta}(q^\rho) s_{\nu/\eta}(q^\rho) \right) s_\mu(\mathbf{x}) s_\nu(\mathbf{y}),$$

where $q = e^{-g_s}$ and g_s the coupling constant, $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$, and the partitions μ, ν encodes boundary conditions of the holomorphic curves we considered in \mathbb{C}^3 .

By the theory of the topological vertex, the generating function of the Gromov-Witten invariants of the resolved conifold with one brane of framing a is given by

$$(17) \quad \tilde{Z}^a(\mathbf{x}) = \sum_{\mu} \left(\sum_{\nu} C_{\mu\nu}^2(a) Q^{|\nu|} C_{\nu}^1 \right) s_{\mu}(\mathbf{x})$$

where $Q = -e^{-t}$ and t is the Kähler parameter of the \mathbb{P}^1 in the resolved conifold, and

$$(18) \quad \begin{aligned} C_{\mu\nu}^2(a) &= q^{\frac{(a+1)\kappa_{\mu}}{2}} \sum_{\eta} s_{\mu^t/\eta}(q^{\rho}) s_{\nu/\eta}(q^{\rho}), \\ C_{\mu}^1 &= s_{\mu}(q^{\rho}). \end{aligned}$$

Let $Z_0 = \sum_{\mu} s_{\mu}(q^{\rho}) Q^{|\mu|} s_{\mu^t}(q^{\rho})$, it is the generating function of the closed Gromov-Witten invariants of the resolved conifold. We are interested in the normalized generating function $Z^a(\mathbf{x}) = \tilde{Z}^a(\mathbf{x})/Z_0$, which is the generating function of the open Gromov-Witten invariants of the famed one-legged resolved conifold.

From now on, we will view Q as a formal variable, Z_0 as a formal power series of Q , and $\tilde{Z}^a(\mathbf{x})$ and $Z^a(\mathbf{x})$ as formal power series of Q with coefficients in the space of symmetric functions (with parameter q).

3.2. Fermionic representation of the generating function. According to the boson-fermion correspondence (13), the element V in the fermionic Fock space \mathcal{F} corresponding to the normalized generating function $Z^a(\mathbf{x})$ defined in the previous subsection is

$$(19) \quad V = \sum_{\mu} \left(\sum_{\nu} C_{\mu\nu}^2(a) Q^{|\nu|} C_{\nu}^1 / Z_0 \right) |\mu\rangle$$

For simplicity, for an integer $m > 0$, we denote $m + 1/2$ by \mathbf{m} and $-m - 1/2$ by $-\mathbf{m}$. The main aim of the present paper is to prove the following

Theorem 3.1. *The element V defined as above is a Bogoliubov transform of the vacuum in \mathcal{F} . In other word, for $m, n \geq 0$, there exist certain coefficients R_{mn} as formal power series of Q , such that*

$$(20) \quad V = \exp \left(\sum_{m, n \geq 0} R_{mn} \psi_{\mathbf{m}} \psi_{-\mathbf{n}}^* \right) |0\rangle$$

where $|0\rangle$ is the vacuum vector in \mathcal{F} . In particular, $Z^a(\mathbf{x})$ is a tau function of the KP hierarchy in the Miwa variables $t_n = \frac{p_n(\mathbf{x})}{n}$.

It seems very difficult to prove Theorem 3.1 by standard Schur calculus. The starting point of the proof here is the formulas of the fermionic representation of the framed one-legged and two-legged topological vertex which were conjectured in [1] and proved in our recent work [3].

Under the boson-fermion correspondence (13), the element $V_1^{(a)} \in \mathcal{F}$ corresponding to the framed one-legged topological vertex $Z_1^a(\mathbf{y})$ is

$$(21) \quad V_1^{(a)} = \sum_{\mu} q^{a\kappa_{\mu}/2} s_{\mu}(q^{\rho}) |\mu\rangle$$

and the element $V_2^{(a_1, a_2)} \in \mathcal{F}_1 \otimes \mathcal{F}_2$ corresponding to the framed two-legged topological vertex $Z_2^{(a_1, a_2)}(\mathbf{x}; \mathbf{y})$ is

$$(22) \quad V_2^{(a_1, a_2)} = \sum_{\mu, \nu} \left(q^{\frac{(a_1+1)\kappa_{\mu} + a_2\kappa_{\nu}}{2}} \sum_{\eta} s_{\mu^t/\eta}(q^{\rho}) s_{\nu/\eta}(q^{\rho}) \right) |\mu\rangle \otimes |\nu\rangle$$

where \mathcal{F}_1 and \mathcal{F}_2 are two copies of \mathcal{F} .

On the two-component fermionic Fock space $\mathcal{F}_1 \otimes \mathcal{F}_2$, define for $i = 1, 2$ operators ψ_r^i and ψ_r^{i*} , $r \in \mathbb{Z} + \frac{1}{2}$. They act on the i -th factor of the tensor product as the operators ψ_r and ψ_r^* respectively, and we use the Koszul sign convention for the anti-commutation relations of these operators, i.e., we set

$$(23) \quad [\psi_r^i, \psi_s^j]_+ = [\psi_r^i, \psi_s^{j*}]_+ = [\psi_r^{i*}, \psi_s^{j*}]_+ = 0$$

for $i \neq j$ and $r, s \in \mathbb{Z} + \frac{1}{2}$.

Note that the charge 0 subspace $(\mathcal{F}_1 \otimes \mathcal{F}_2)^{(0)}$ of $\mathcal{F}_1 \otimes \mathcal{F}_2$ has a natural decomposition as

$$(24) \quad (\mathcal{F}_1 \otimes \mathcal{F}_2)^{(0)} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{F}_1^{(n)} \otimes \mathcal{F}_2^{(-n)})$$

For an element $W \in (\mathcal{F}_1 \otimes \mathcal{F}_2)^{(0)}$, we denote by W^0 the projection of W to the component $\mathcal{F}_1^{(0)} \otimes \mathcal{F}_2^{(0)}$ with respect to this decomposition.

The main result proved in [3] is the following

Theorem 3.2. *For $m, n \geq 0$ and $i, j = 1, 2$, there exist coefficients $A_{mn} = A_{mn}(a)$ and $A_{mn}^{ij} = A_{mn}^{ij}(a_1, a_2)$ such that*

$$(25) \quad \begin{aligned} V_1^{(a)} &= \exp\left(\sum_{m, n \geq 0} A_{mn}(a) \psi_{\mathbf{m}} \psi_{-\mathbf{n}}^*\right) |0\rangle \\ V_2^{(a_1, a_2)} &= \left(\exp\left(\sum_{i, j=1, 2} \sum_{m, n \geq 0} A_{mn}^{ij}(a_1, a_2) \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}\right) |0_{12}\rangle \right)^0 \end{aligned}$$

where $|0_{12}\rangle = |0_1\rangle \otimes |0_2\rangle$ is the vacuum vector in $\mathcal{F}_1 \otimes \mathcal{F}_2$.

The coefficients A_{mn}^{ij} and A_{mn} in Theorem 3.2 can be given explicitly (see [1][3]):

$$\begin{aligned} A_{mn}(a) &= (-1)^n q^{\frac{(2a+1)(m(m+1)-n(n+1))}{4}} \frac{1}{[m+n+1][m]![n]!}, \\ A_{mn}^{11}(a_1, a_2) &= (-1)^n q^{\frac{(2a_1+1)m(m+1)-(2a_2+1)n(n+1)}{4} + \frac{1}{6}} \sum_{l=0}^{\min(m,n)} \frac{q^{\frac{1}{2}(l+1)(m+n-l)}}{[m-l]![n-l]!}, \\ A_{mn}^{22}(a_1, a_2) &= (-1)^n q^{\frac{(2a_2+1)m(m+1)-(2a_1+1)n(n+1)}{4} + \frac{1}{6}} \sum_{l=0}^{\min(m,n)} \frac{q^{\frac{1}{2}(l+1)(m+n-l)}}{[m-l]![n-l]!}, \\ A_{mn}^{12}(a_1, a_2) &= (-1)^n q^{\frac{(2a_1+1)m(m+1)-(2a_2+1)n(n+1)}{4} + \frac{1}{6}} \sum_{l=0}^{\min(m,n)} \frac{q^{\frac{1}{2}(l+1)(m+n-l)}}{[m-l]![n-l]!}, \\ A_{mn}^{21}(a_1, a_2) &= -(-1)^n q^{\frac{(2a_2+1)m(m+1)-(2a_1+1)n(n+1)}{4} - \frac{1}{6}} \sum_{l=0}^{\min(m,n)} \frac{q^{-\frac{1}{2}(l+1)(m+n-l)}}{[m-l]![n-l]!}. \end{aligned}$$

Let $\tilde{V} = Z_0 V \in \mathcal{F}$, define

$$(26) \quad V'_1 = \sum_{\mu} |Q|^{|\mu|} s_{\mu^t}(q^\rho) |\mu\rangle$$

as a formal power series of Q with coefficients in \mathcal{F} . We view \tilde{V} as an element in \mathcal{F}_1 and V'_1 an element in \mathcal{F}_2 . By the definition of the inner product on \mathcal{F} and (17), it is clear that

$$(27) \quad \tilde{V} = (V_2^{(a,0)}, V'_1)$$

where the inner product is taken on the \mathcal{F}_2 component.

The following lemma, which shows that V'_1 is also a Bogoliubov transform of the fermionic vacuum, will be used in our proof of Theorem 3.1.

Lemma 3.3. *The element V'_1 is a Bogoliubov transform of the fermionic vacuum, i.e., it can be represented as*

$$(28) \quad V'_1 = \exp\left(\sum_{m,n \geq 0} A'_{mn} \psi_{\mathbf{m}} \psi_{-\mathbf{n}}^*\right) |0\rangle,$$

where the coefficients

$$(29) \quad A'_{mn} = Q^{m+n+1} q^{\frac{1}{2}(n(n+1)-m(m+1))} A_{mn}(0).$$

Proof. Let $\mu = (m_1, \dots, m_k | n_1, \dots, n_k)$ be a partition in Frobenius notation. Then $\kappa_{\mu^t} = -\kappa_{\mu}$ by Lemma 2.1, and hence $s_{\mu^t}(q^\rho) = q^{-\frac{\kappa_{\mu}}{2}} s_{\mu}(q^\rho)$ by Proposition 2.2. By (9), (11) and Theorem 3.2, one can show that $s_{\mu}(q^\rho) = \det(A_{m_i n_j}(0))_{k \times k}$. Note that $|\mu| = \sum_{i=1}^k (m_i + n_i + 1)$. If we take $A'_{mn} = Q^{m+n+1} q^{\frac{1}{2}(n(n+1)-m(m+1))} A_{mn}(0)$, then we have $Q^{|\mu|} s_{\mu^t}(q^\rho) = \det(A'_{m_i n_j})_{k \times k}$, and hence $V'_1 = \exp(\sum_{m,n \geq 0} A'_{mn} \psi_{\mathbf{m}} \psi_{-\mathbf{n}}^*) |0\rangle$. \square

By (27), Theorem 3.2 and Lemma 3.3, it is clear that Theorem 3.1 is a direct corollary of Theorem 4.1 that we will prove in §4.

Provided Theorem 3.1, it is easy to determine the coefficients R_{mn} appearing in (20). They are given by

$$(30) \quad R_{mn} = \sum_{\nu} C_{(m|n)\nu}^2(a) Q^{|\nu|} s_{\nu^t}(q^\rho) / Z_0.$$

But it is not our aim here to study various simple forms of R_{mn} .

4. GLUING OF BOGOLIUBOV TRANSFORMS

We have mentioned the notion of (one-component) Bogoliubov transform in the previous section. Recall that a vector V in the 2-fold fermionic Fock space $\mathcal{F}_1 \otimes \mathcal{F}_2$ is called a Bogoliubov transform (of the fermionic vacuum) if it is given by the vacuum in $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n$ acted upon by an exponential of a quadratic expression of fermionic creators. In other word, it can be represented as

$$(31) \quad V = \exp\left(\sum_{i,j=1}^2 \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}\right) |0\rangle$$

where $|0\rangle$ is the vacuum in $\mathcal{F}_1 \otimes \mathcal{F}_2$ and A_{mn}^{ij} are certain coefficients possibly with parameters.

In this section, we study properties of fermionic states which are constructed by gluing (to be defined later) one-component and two-component Bogoliubov transforms. If we glue an arbitrary one-component Bogoliubov transform and an arbitrary two-component Bogoliubov transform, then we get a state in the fermionic Fock space \mathcal{F} . Our aim here is to prove that the state we get is also a Bogoliubov transform of the fermionic vacuum. In particular, it is a tau function of the KP hierarchy.

Let

$$(32) \quad V_1 = \exp\left(\sum_{i,j=1,2} \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}\right) |0_{12}\rangle$$

be a two-component Bogoliubov transform in $\mathcal{F}_1 \otimes \mathcal{F}_2$ and

$$(33) \quad V_2 = \exp\left(\sum_{m,n \geq 0} A_{mn} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}\right) |0_2\rangle$$

be an one-component Bogoliubov transform in \mathcal{F}_2 , where A_{mn}^{ij} and A_{mn} are arbitrary coefficients maybe with parameters.

Define

$$(34) \quad \tilde{V}_2 = \exp\left(\sum_{m,n \geq 0} Q^{m+n+1} A_{mn} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}\right) |0_2\rangle$$

where Q is a formal variable and \tilde{V}_2 is viewed as a formal power series of Q with coefficients in \mathcal{F}_2 .

The following inner product

$$(35) \quad \begin{aligned} V_0 &= \left(\exp\left(\sum_{m,n \geq 0} A_{mn}^{22} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}\right) |0_2\rangle, \exp\left(\sum_{m,n \geq 0} Q^{m+n+1} A_{mn} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}\right) |0_2\rangle \right) \\ &= \langle 0_2 | \exp\left(\sum_{m,n \geq 0} Q^{m+n+1} A_{mn} \psi_{-\mathbf{n}}^2 \psi_{\mathbf{m}}^{2*}\right) \exp\left(\sum_{m,n \geq 0} A_{mn}^{22} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}\right) |0_2\rangle \end{aligned}$$

is well defined as a formal power series of the formal variable Q .

Define $\tilde{V} = (V_1, \tilde{V}_2)$, where the inner product is taken on the \mathcal{F}_2 component. Then \tilde{V} is a formal power series of Q with coefficients in \mathcal{F}_1 .

Theorem 4.1. *Let \tilde{V} and V_0 be as above, then the formal power series $V = \tilde{V}/V_0$ of Q with coefficients in \mathcal{F}_1 is a Bogoliubov transform of the fermionic vacuum $|0_1\rangle \in \mathcal{F}_1$, i.e., for $m, n \geq 0$, there exist formal power series R_{mn} of Q , such that*

$$(36) \quad V = \exp\left(\sum_{m,n \geq 0} R_{mn} \psi_{\mathbf{m}}^1 \psi_{-\mathbf{n}}^{1*}\right) |0_1\rangle.$$

From the proof of this theorem, we will see why V_0 appear naturally as a factor of \tilde{V} . The following lemma, which is well-known in Lie theory, will be used in the proof of Theorem 4.1.

Lemma 4.2. *(See e.g [6]) Let A and B be two linear operators on a vector space H . Assume both e^A and e^B make sense. If the commutator $[A, B] = AB - BA$ commutes with both A and B . Then*

$$(37) \quad e^A e^B = e^{[A, B]} e^B e^A.$$

Proof. (Proof of Theorem 4.1) Recall that

$$\tilde{V} = (\exp\left(\sum_{i,j=1,2} \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}\right) |0_{12}\rangle, \exp\left(\sum_{m,n \geq 0} Q^{m+n+1} A_{mn} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}\right) |0_2\rangle).$$

To make the notations simpler, let

$$(38) \quad \mathcal{A}^{ij} = \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*},$$

$$(39) \quad \mathcal{A} = \sum_{m,n \geq 0} Q^{m+n+1} A_{mn} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}.$$

Then one can rewrite \tilde{V} as follows:

$$(40) \quad \tilde{V} = \exp \mathcal{A}^{11} \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{21}) \exp(\mathcal{A}^{12}) \exp(\mathcal{A}^{22}) | 0_2 \rangle | 0_1 \rangle$$

As usual, our strategy here is to move the annihilators to the right using the anti-commutation relations (9). By (9) and (23), one has

$$(41) \quad [\mathcal{A}^*, \mathcal{A}^{21}] = \mathcal{B}^{21} = \sum_{m,n \geq 0} B_{mn}^{21} \psi_{-\mathbf{m}}^2 \psi_{-\mathbf{n}}^{1*},$$

where

$$(42) \quad B_{mn}^{21} = \sum_{r \geq 0} Q^{r+m+1} A_{rm} A_{rn}^{21}$$

are formal power series of Q which are divisible by Q . Note that the right-hand side commutes with both \mathcal{A}^* and \mathcal{A}^{21} , hence by Lemma 4.2 we get

$$\exp(\mathcal{A}^*) \exp(\mathcal{A}^{21}) = \exp(\mathcal{A}^{21}) \exp(\mathcal{A}^*) \exp(\mathcal{B}^{21}).$$

By the same method, one can show that

$$(43) \quad \exp(\mathcal{B}^{21}) \exp(\mathcal{A}^{12}) = \exp(\mathcal{A}^{21,12}) \exp(\mathcal{A}^{12}) \exp(\mathcal{B}^{21}),$$

where

$$(44) \quad \mathcal{A}^{21,12} = [\mathcal{B}^{21}, \mathcal{A}^{12}] = - \sum_{m,n \geq 0} \sum_{r \geq 0} A_{mr}^{12} B_{rn}^{21} \psi_{\mathbf{m}}^1 \psi_{-\mathbf{n}}^{1*},$$

and

$$(45) \quad \exp(\mathcal{B}^{21}) \exp(\mathcal{A}^{22}) = \exp(\mathcal{A}^{21,1}) \exp(\mathcal{A}^{22}) \exp(\mathcal{B}^{21}),$$

where

$$(46) \quad \mathcal{A}^{21,1} = [\mathcal{B}^{21}, \mathcal{A}^{22}] = - \sum_{m,n \geq 0} \sum_{r \geq 0} A_{mr}^{22} B_{rn}^{21} \psi_m^2 \psi_{-n}^{1*}.$$

Now we have

$$\tilde{V} = \exp(\mathcal{A}^{11} + \mathcal{A}^{2112,1}) \langle 0_2 | \exp(\mathcal{A}^{21}) \exp(\mathcal{A}^*) \exp(\mathcal{A}^{12}) \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{21,1}) \exp(\mathcal{B}^{21}) | 0_2 \rangle | 0_1 \rangle.$$

Note $\mathcal{B}^{21}|0_2\rangle = 0$ and $\langle 0_2 | \mathcal{A}^{21} = 0$, we have

$$(47) \quad \tilde{V} = \exp(\mathcal{A}^{11} + \mathcal{A}^{2112,1}) \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{12}) \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{21,1}) | 0_2 \rangle | 0_1 \rangle.$$

Similarly, one can show that

$$(48) \quad \exp(\mathcal{A}^*) \exp(\mathcal{A}^{12}) = \exp(\mathcal{A}^{12}) \exp(\mathcal{A}^*) \exp(\mathcal{B}^{12}),$$

where

$$\mathcal{B}^{12} = [\mathcal{A}^*, \mathcal{A}^{12}] = \sum_{m,n \geq 0} B_{mn}^{12} \psi_m^1 \psi_n^{2*}$$

which commutes with both \mathcal{A}^* and \mathcal{A}^{12} , where

$$(49) \quad B_{mn}^{12} = - \sum_{r \geq 0} A_{mr}^{12} Q^{n+r+1} A_{nr}$$

are formal power series of Q which are divisible by Q .

$$(50) \quad \exp(\mathcal{B}^{12}) \exp(\mathcal{A}^{22}) = \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{12,1}) \exp(\mathcal{B}^{12}),$$

where

$$(51) \quad \mathcal{A}^{12,1} = [\mathcal{B}^{12}, \mathcal{A}^{22}] = \sum_{m,n \geq 0} \sum_{r \geq 0} B_{mr}^{12} A_{rn}^{22} \psi_m^1 \psi_{-n}^{2*},$$

and

$$(52) \quad \exp(\mathcal{B}^{12}) \exp(\mathcal{A}^{21,1}) = \exp(\mathcal{A}^{1221,1}) \exp(\mathcal{A}^{21,1}) \exp(\mathcal{B}^{12}),$$

where

$$(53) \quad \mathcal{A}^{1221,1} = [\mathcal{B}^{12}, \mathcal{A}^{21,1}] = \sum_{m,n \geq 0} \left(\sum_{r \geq 0} B_{mr}^{12} A_{rn}^{21,1} \right) \psi_m^1 \psi_{-n}^{1*}$$

commutes with both \mathcal{B}^{12} and $\mathcal{A}^{21,1}$. Because $\langle 0_2 | \mathcal{A}^{12} = 0$ and $\mathcal{B}^{12}|0_2\rangle = 0$,

$$\begin{aligned} & \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{12}) \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{21,1}) | 0_2 \rangle \\ &= \langle 0_2 | \exp(\mathcal{A}^{12}) \exp(\mathcal{A}^*) \exp(\mathcal{B}^{12}) \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{21,1}) | 0_2 \rangle \\ &= \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{12,1}) \exp(\mathcal{A}^{1221,1}) \exp(\mathcal{A}^{21,1}) \exp(\mathcal{B}^{12}) | 0_2 \rangle \\ &= \exp(\mathcal{A}^{1221,1}) \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{12,1}) \exp(\mathcal{A}^{21,1}) | 0_2 \rangle. \end{aligned}$$

Because the operators \mathcal{A}^{22} , $\mathcal{A}^{12,1}$ and $\mathcal{A}^{21,1}$ commute with each other, we now have

$$\tilde{V} = \exp(\mathcal{A}^{11} + \mathcal{A}^{2112,1} + \mathcal{A}^{1221,1}) \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{21,1}) \exp(\mathcal{A}^{12,1}) \exp(\mathcal{A}^{22}) | 0_2 \rangle | 0_1 \rangle.$$

Recall $\mathcal{A}^{2112,1}$, $\mathcal{A}^{12,1}$, $\mathcal{A}^{21,1}$ are divisible by Q , and $\mathcal{A}^{1221,1}$ is divisible by Q^2 . By repeating the above procedure N -times one gets:

$$\begin{aligned} \tilde{V} &= \exp(\mathcal{A}^{11} + \sum_{j=1}^N (\mathcal{A}^{2112,j} + \mathcal{A}^{1221,j})) \\ &\quad \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{21,N}) \exp(\mathcal{A}^{12,N}) \exp(\mathcal{A}^{22}) | 0_2 \rangle | 0_1 \rangle, \end{aligned}$$

where $\mathcal{A}^{2121,j}, \mathcal{A}^{12,j}, \mathcal{A}^{21,j}$ and $\mathcal{A}^{1221,j}$ is divisible by Q^j . Therefore, by taking $N \rightarrow \infty$,

$$(54) \quad \tilde{V} = \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{22}) | 0_2 \rangle \cdot \exp(\mathcal{A}^{11} + \sum_{j=1}^{\infty} (\mathcal{A}^{2112,j} + \mathcal{A}^{1221,j}) | 0_1 \rangle.$$

This completes the proof of Theorem 4.1. \square

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